Information & Security
Published since 1998

J. Dezert & A. Tchamova
vol. 52 (2022): 37-51
https://doi.org/10.11610/isij.5202
ISSN 0861-5160 (print), ISSN 1314-2119 (online)

Research Article

On Inequalities Bounding Imprecision and Nonspecificity Measures of Uncertainty

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ABSTRACT:

This article presents proof of two inequalities about two measures of uncertainty of basic belief assignments, called respectively Imprecision measure and U-uncertainty measure, that have been introduced by Dubois and Prade in the 1980s. These inequalities have been considered as obvious by the authors, but to prove them rigorously requires some effort, as demonstrated in this article.

ARTICLE INFO:

RECEIVED: 06 JAN 2022
REVISED: 26 JAN 2022
ONLINE: 02 FEB 2022

KEYWORDS:

belief functions, measures of uncertainty, convex combination, inequalities

1. Introduction

This article presents two mathematical proofs of inequalities about two measures of uncertainty of basic belief assignments, called respectively the Imprecision and the U-uncertainty (or non-specificity) that have been introduced by Dubois and Prade.[1,2,3] We recall that a Basic Belief Assignment (BBA) m defined on the power set 2^{Θ} of the finite frame of discernment (FoD)

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$\Theta=\{\theta_1,\theta_2,...,\theta_n\}$ is a mapping $m(\cdot):2^\Theta \to [0,1]$ such that $m(\varnothing)=0$ and $\sum_{X\subseteq\Theta} m(X)=1$. This type of mapping has been introduced by Shafer.[4] The cardinality of the FoD is $|\Theta|=n$. The measures of imprecision $I(m)$ and of nonspecificity $U(m)$ are respectively defined by

$$I(m)=\sum_{X\subseteq\Theta} m(X)|X| = \sum_{X_i\in 2^\Theta} m(X_i)|X_i| \tag{1}$$

$$U(m)=\sum_{X\subseteq\Theta} m(X)\log(|X|) = \sum_{X_i\in 2^\Theta} m(X_i)\log(|X_i|), \tag{2}$$

where X_i is the i-th element of the power set 2^Θ of the FoD Θ and $|X_i|$ is its cardinality. By convention, and without loss of generality, we will take $X_1=\varnothing$ (the empty set), and $X_{2^n}=\Theta$. The integer index i varies from 1 to $2^n=2^{|\Theta|}$.

m_v is the vacuous BBA defined by $m_v(X)=1$ If $X=\Theta$ and $m_v(X)=0$ for all elements $X\neq\Theta$ of 2^Θ. This vacuous BBA m_v characterizes a full ignorant source of evidence.

In the next sections we prove that for any BBA $m\neq m_v$ defined on 2^Θ the two following inequalities hold

$$I(m)<I(m_v) \tag{3}$$

and

$$U(m)<U(m_v) \tag{4}$$

We will prove these two inequalities in two ways: 1) by a direct application of the Theorem of convex combination (see Theorem 1), and 2) by a direct calculation from the mathematical definitions of $I(m)$ and $U(m)$ measures of uncertainty.

For proving these inequalities, we first recall that a convex combination, denoted by s_n, of n values $\{z_i, i=1,2,...,n\}$ is a linear combination of the form

$$s_n=\sum_{i=1}^n w_i z_i \tag{5}$$

where $w_i\in[0,1]$ is the weight of the value z_i such that $\sum_{i=1}^n w_i=1$.

In the appendix, we prove the following useful theorem that will help us to prove the inequalities (3) and (4) in the next sections.

Theorem 1: Let $s_n = \sum_{i=1}^{n} w_i z_i$ be a convex combination of n values $z_1, z_2, ..., z_n$ with normalized weights $w_1, w_2, ..., w_n$, where $w_i \in [0,1]$. Then, we have

$$\min\{z_i \in Z\} \leq s_n \leq \max\{z_i \in Z\} \quad (6)$$

where $Z = \{z_i \in \{z_1, z_2, ..., z_n\} \mid w_i > 0\}$.

Proof of Theorem 1: see appendix.

2. Proofs that $I(m) < I(m_v)$ if $m \neq m_v$

2.1. First proof: using the theorem of convex combination

The proof of inequality (3) is a direct application of Theorem 1 when working with $2^n = 2^{|\Theta|}$ values $z_i = |X_i|$ and weights $w_i = m(X_i)$. We recall that integer index i spans $\{1,2,...,2^n\}$ and that $w_1 = m(X_1) = m(\emptyset) = 0$ for any BBA m (by definition of m). Therefore, one has always at least one weight (i.e. w_1) among all 2^n weights equals zero, which justifies the use of Theorem 1, rather than Theorem 2 of appendix.

The imprecision measure $I(m)$ can also be expressed as $I(m) = \sum_{i=1}^{2^n} m(X_i)|X_i|$ because

$$\sum_{X_i \in 2^{\Theta}} m(X_i)|X_i| = \sum_{i=1}^{2^n} m(X_i)|X_i|$$

Based on Theorem 1, we have

$$\begin{aligned} &\min\left\{|X_i| \in \left\{|X_1|, |X_2|, ..., |X_{2^n}|\right\} \mid m(X_i) > 0\right\} \\ &\leq \sum_{i=1}^{2^n} m(X_i)|X_i| \leq \\ &\max\left\{|X_i| \in \left\{|X_1|, |X_2|, ..., |X_{2^n}|\right\} \mid m(X_i) > 0\right\} \end{aligned} \quad (7)$$

The upper bound of inequality (7) is always lower than $|\Theta| = n$ if $m \neq m_v$ and it is equal to $|\Theta| = n$ when $m = m_v$.

Therefore, one has

$$\sum_{i=1}^{2^n} m(X_i)|X_i| < |\Theta| \qquad (8)$$

and because $l(m_v) = m_v(\Theta)\cdot|\Theta| = 1\cdot|\Theta| = |\Theta|$, one sees that the valid inequality (8) is the same as

$$l(m) < l(m_v) \qquad (9)$$

which completes the proof of the inequality (3).

2.2. Second proof: using direct calculation

First, we note that

$$l(m_v) = m(\Theta)\cdot|\Theta| = 1\cdot n = n$$

Because m is a (normalized) BBA [4] such that $m(\emptyset)=0$ and $\sum_{X\subseteq\Theta} m(X) = 1$, one has

$$m(\Theta) + \sum_{X\subset\Theta} m(X) = 1$$

Or, equivalently

$$m(\Theta) = 1 - \sum_{X\subset\Theta} m(X)$$

Therefore,

$$n\cdot m(\Theta) = n\cdot\left[1 - \sum_{X\subset\Theta} m(X)\right]$$

The expression of $l(m)$ can be decomposed as

$$I(m)=\sum_{X\subseteq\Theta}m(X)|X|=m(\Theta)\cdot|\Theta|+\sum_{X\subset\Theta}m(X)|X|$$
$$=n\cdot m(\Theta)+\sum_{X\subset\Theta}m(X)|X|=n\cdot\left[1-\sum_{X\subset\Theta}m(X)\right]+\sum_{X\subset\Theta}m(X)|X|$$
$$=n-\left[n\cdot\sum_{X\subset\Theta}m(X)-\sum_{X\subset\Theta}m(X)|X|\right]$$

To prove that $I(m)<I(m_v)$ is equivalent to prove that

$$n-\left[n\cdot\sum_{X\subset\Theta}m(X)-\sum_{X\subset\Theta}m(X)|X|\right]<n$$

or to prove

$$n\cdot\sum_{X\subset\Theta}m(X)-\sum_{X\subset\Theta}m(X)|X|>0 \tag{10}$$

We can express $\sum_{X\subset\Theta}m(X)|X|$ as

$$\sum_{X\subset\Theta}m(X)|X|=\sum_{X\subset\Theta\,s.t.|X|=1}m(X)\cdot 1+\sum_{X\subset\Theta\,s.t.|X|=2}m(X)\cdot 2+\sum_{X\subset\Theta\,s.t.|X|=3}m(X)\cdot 3$$
$$+\ldots+\sum_{X\subset\Theta\,s.t.|X|=n-1}m(X)\cdot(n-1)$$

That is

$$\sum_{X\subset\Theta}m(X)|X|=\sum_{X\subset\Theta\,s.t.|X|=1}m(X)+2\cdot\sum_{X\subset\Theta\,s.t.|X|=2}m(X)+3\cdot\sum_{X\subset\Theta\,s.t.|X|=3}m(X)$$
$$+\ldots+(n-1)\cdot\sum_{X\subset\Theta\,s.t.|X|=n-1}m(X)$$

which can be rewritten as

$$\sum_{X\subset\Theta} m(X)|X| = \sum_{X\subset\Theta \text{ s.t.} |X|=1} m(X)$$
$$+ \sum_{X\subset\Theta \text{ s.t.} |X|=2} m(X) + \sum_{X\subset\Theta \text{ s.t.} |X|=2} m(X)$$
$$\sum_{X\subset\Theta \text{ s.t.} |X|=3} m(X) + 2\cdot \sum_{X\subset\Theta \text{ s.t.} |X|=3} m(X)$$
$$+ \dots\dots\dots\dots$$
$$+ \sum_{X\subset\Theta \text{ s.t.} |X|=n-1} m(X) + (n-2)\cdot \sum_{X\subset\Theta \text{ s.t.} |X|=n-1} m(X)$$

Or equivalently

$$\sum_{X\subset\Theta} m(X)|X| = \sum_{X\subset\Theta} m(X) + \sum_{X\subset\Theta \text{ s.t.} |X|=2} m(X) +$$
$$+ 2\cdot \sum_{X\subset\Theta \text{ s.t.} |X|=3} m(X) +$$
$$+ \dots$$
$$+ (n-2)\cdot \sum_{X\subset\Theta \text{ s.t.} |X|=n-1} m(X) +$$

Then, for the left hand side of the inequality (10) we obtain the following expression

$$n\cdot \sum_{X\subset\Theta} m(X) - \sum_{X\subset\Theta} m(X)|X| = (n-1)\cdot \sum_{X\subset\Theta \text{ s.t.} |X|=1} m(X)$$
$$+ (n-2)\cdot \sum_{X\subset\Theta \text{ s.t.} |X|=2} m(X)$$
$$+ (n-3)\cdot \sum_{X\subset\Theta \text{ s.t.} |X|=3} m(X)$$
$$+ \dots\dots$$
$$+ (n-(n-1))\cdot \sum_{X\subset\Theta \text{ s.t.} |X|=n-1} m(X)$$

The right hand side of the previous expression is strictly positive, that is

$$(n-1)\cdot \sum_{X\subset\Theta \text{ s.t.} |X|=1} m(X) + (n-2)\cdot \sum_{X\subset\Theta \text{ s.t.} |X|=2} m(X) + (n-3)\cdot \sum_{X\subset\Theta \text{ s.t.} |X|=3} m(X)$$
$$+ \cdots + (n-(n-1))\cdot \sum_{X\subset\Theta \text{ s.t.} |X|=n-1} m(X) > 0$$

because $n>1$ (the FoD has more than one hypothesis inside), and also because there is at least one element $X \subset \Theta$ for which $m(X)>0$ when $m \neq m_v$.

Then, we obtain

$$n \cdot \sum_{X \subset \Theta} m(X) - \sum_{X \subset \Theta} m(X)|X| > 0 ,$$

which completes our second proof of (3) by a direct calculation.

3. Proofs that $U(m) < U(m_v)$ if $m \neq m_v$

3.1. First proof: using the theorem of convex combination

The proof of inequality $U(m) < U(m_v)$ is similar to the proof of $I(m) < I(m_v)$ by replacing values $|X_i|$ by $\log(|X_i|)$, and by taking $m(X_1)\log(|X_1|) = m(\emptyset)\log(|\emptyset|) = 0 \cdot \log(0) = 0$, which is easily justified by continuity because $x\log(x) \to 0$ as $x \to 0$. More precisely, we can express $U(m)$ as

$$U(m) = m(\emptyset)\log(|\emptyset|) + \sum_{X_i \in 2^\Theta \setminus \{\emptyset\}} m(X_i)\log(|X_i|) = \sum_{i=2}^{2^n} m(X_i)\log(|X_i|)$$

Based on Theorem 1, we have

$$\min\left\{\log(|X_i|) \in \left\{\log(|X_2|),\dots,\log(|X_{2^n}|)\right\} | m(X_i) > 0\right\}$$
$$\leq \sum_{i=2}^{2^n} m(X_i)\log(|X_i|) \leq \tag{11}$$
$$\max\left\{\log(|X_i|) \in \left\{\log(|X_2|),\dots,\log(|X_{2^n}|)\right\} | m(X_i) > 0\right\}$$

Because $\log(.)$ is a continuous increasing function, the upper bound of the previous inequality is always lower than $\log(|\Theta|) = \log(n)$ when $m \neq m_v$. Therefore,

$$\sum_{i=2}^{2^n} m(X_i)\log(|X_i|) < \log(|\Theta|) \quad (12)$$

and because $U(m_v) = m_v(\Theta)\cdot\log(|\Theta|) = 1\cdot\log(|\Theta|) = \log(|\Theta|)$, one sees that the valid inequality (12) is the same as

$$U(m) < U(m_v) \quad (13)$$

which completes the proof of the inequality (4).

3.2. Second proof: using direct calculation

We prove the inequality $U(m) < U(m_v)$ similarly to our second proof for $I(m) < I(m_v)$ by replacing values $|X_i|$ by $\log(|X_i|)$. We note that

$$U(m_v) = m(\Theta)\cdot\log(|\Theta|) = 1\cdot\log(n) = \log(n)$$

Because m is a (normalized) BBA [4] such that $m(\varnothing) = 0$ and $\sum_{X\subseteq\Theta} m(X) = 1$, one has

$$m(\Theta) + \sum_{X\subset\Theta} m(X) = 1$$

Or, equivalently

$$m(\Theta) = 1 - \sum_{X\subset\Theta} m(X)$$

Therefore,

$$\log(n)\cdot m(\Theta) = \log(n)\cdot\left[1 - \sum_{X\subset\Theta} m(X)\right]$$

The expression of $U(m)$ can be decomposed as

$$\begin{aligned}U(m) &= \sum_{X\subseteq\Theta} m(X)\log\left(|X|\right) = \\ &= m(\Theta)\cdot\log(n) + \sum_{X\subset\Theta} m(X)\log(|X|) \\ &= \log(n)\cdot\left[1-\sum_{X\subset\Theta} m(X)\right] + \sum_{X\subset\Theta} m(X)\log(|X|) \\ &= \log(n) - \left[\log(n)\sum_{X\subset\Theta} m(X) - \sum_{X\subset\Theta} m(X)\log(|X|)\right]\end{aligned}$$

To prove that $U(m) < U(m_v)$ is equivalent to prove that

$$\log(n) - \left[\log(n)\cdot\sum_{X\subset\Theta} m(X) - \sum_{X\subset\Theta} m(X)\log(|X|)\right] < \log(n)$$

or to prove

$$\log(n)\cdot\sum_{X\subset\Theta} m(X) - \sum_{X\subset\Theta} m(X)\log(|X|) > 0 \tag{14}$$

We can express $\sum_{X\subset\Theta} m(X)\log\left(|X|\right)$ as

$$\begin{aligned}\sum_{X\subset\Theta} m(X)\log\left(|X|\right) = &\sum_{X\subset\Theta\,s.t.\,|X|=1} m(X)\cdot\log(1) \\ &+ \sum_{X\subset\Theta\,s.t.\,|X|=2} m(X)\cdot\log(2) \\ &+ \sum_{X\subset\Theta\,s.t.\,|X|=3} m(X)\cdot\log(3) \\ &+ \ldots\ldots.. \\ &+ \sum_{X\subset\Theta\,s.t.\,|X|=n-1} m(X)\cdot\log(n-1)\end{aligned}$$

Then for the left hand side of inequality (14) we obtain:

$$\log(n)\cdot\sum_{X\subset\Theta} m(X)-\sum_{X\subset\Theta} m(X)\log(|X|)= [\log(n)-\log(1)]\cdot\sum_{X\subset\Theta \text{ s.t.} |X|=1} m(X)$$
$$+[\log(n)-\log(2)]\cdot\sum_{X\subset\Theta \text{ s.t.} |X|=2} m(X)$$
$$+[\log(n)-\log(3)]\cdot\sum_{X\subset\Theta \text{ s.t.} |X|=3} m(X)$$
$$+\dots\dots$$
$$+[\log(n)-\log(n-1)]\cdot\sum_{X\subset\Theta \text{ s.t.} |X|=n-1} m(X)$$

Because $\log(1)=0$, the equation above can be rewritten as

$$\log(n)\cdot\sum_{X\subset\Theta} m(X)-\sum_{X\subset\Theta} m(X)\log(|X|)=\log(n)\cdot\sum_{X\subset\Theta \text{ s.t.} |X|=1} m(X)$$
$$+[\log(n)-\log(2)]\cdot\sum_{X\subset\Theta \text{ s.t.} |X|=2} m(X)$$
$$+[\log(n)-\log(3)]\cdot\sum_{X\subset\Theta \text{ s.t.} |X|=3} m(X)$$
$$+\dots\dots$$
$$+[\log(n)-\log(n-1)]\cdot\sum_{X\subset\Theta \text{ s.t.} |X|=n-1} m(X)$$

Because $n>1$, and because $\log(.)$ is an increasing function one always has $\log(n)>0, [\log(n)-\log(2)]>0,\dots,[\log(n)-\log(n-1)]>0$. Because there is at least one element $X\subset\Theta$ for which $m(X)>0$ when $m\neq m_v$, we can conclude that

$$\log(n)\sum_{X\subset\Theta} m(X)-\sum_{X\subset\Theta} m(X)\log(|X|)>0$$

which completes our second proof of (4) by a direct calculation.

4. Conclusion

In this paper we have proved that the imprecision measure $I(m)$ is always lower than $I(m_v)=|\Theta|$, and its U - uncertainty (also known as non-specificity) measure $U(m)$ is always lower than $U(m_v)=\log(|\Theta|)$ for any non-vacuous BBA m. The proofs presented in this paper have been obtained by two different ways: by the theorem of convex combination, and by direct calculation from the mathematical definitions for $I(m), I(m_v), U(m)$, and $U(m_v)$. We have shown that the use of the theorem of convex combination provides an elegant and shorter proof of the inequalities. This theorem will be helpful to evaluate the lower and upper bounds of any measures of uncertainty of a BBA that would be based on any convex combination of mass values (chosen as weighting factors) and real values committed to each element of the power set of the frame of discernment.

Appendix

Before proving Theorem 1, we need to establish at first the following theorem.

Theorem 2: Let $s_n = \sum_{i=1}^{n} w_i z_i$ be a convex combination of n values $z_1, z_2, \ldots, z_n$ with strictly positive normalized weights $w_1, w_2, \ldots, w_n$. Then, we have

$$\min\{z_1, \ldots, z_n\} \leq s_n \leq \max\{z_1, \ldots, z_n\} \tag{15}$$

The proof of Theorem 2 is done by induction.

Proof of Theorem 2:

For $n=1$, one has only one value z_1 with weight $w_1 = 1$. Hence $s_1 = w_1 z_1 = z_1$, $\min\{z_1\} = z_1$, and $\max\{z_1\} = z_1$. Therefore, $\min\{z_1\} = s_1 = \max\{z_1\}$, which is a special case of the inequality (15). Consequently, the inequality (15) is valid for $n=1$.

For $n=2$, one has two values $\{z_1, z_2\}$ with (strictly) positive weights $\{w_1, w_2\}$ and $s_2 = w_1 z_1 + w_2 z_2$.

1) if $z_1 = z_2$, then $s_2 = w_1 z_1 + w_2 z_2 = w_1 z_1 + w_2 z_1 = (w_1 + w_2) z_1 = z_1 = z_2$. Hence one has $\min\{z_1, z_2\} = z_1 = z_2$ and $\max\{z_1, z_2\} = z_1 = z_2$. Therefore, one gets $\min\{z_1, z_2\} = s_2 = \max\{z_1, z_2\}$, which a special case of the inequality (15) is for $n=2$.
2) If $z_1 \neq z_2$, then two sub-cases are possible:

a) Case 1: if $z_1 < z_2$, then $\min\{z_1, z_2\} = z_1$ and $\max\{z_1, z_2\} = z_2$. We have

$$\begin{aligned} s_2 &= w_1 z_1 + w_2 z_2 = (z_1 - z_1) + w_1 z_1 + w_2 z_2 \\ &= z_1 - (1 - w_1) z_1 + w_2 z_2 = z_1 - w_2 z_1 + w_2 z_2 \\ &= z_1 + w_2 (z_2 - z_1) \geq \min\{z_1, z_2\} \end{aligned}$$

This last inequality comes from the fact that $w_2 \geq 0$, and $z_2 - z_1 \geq 0$ because $\min\{z_1, z_2\} = z_1$. So we have proved $\min\{z_1, z_2\} \leq s_2$.

Because $w_2 \in [0,1]$, we have $w_2 (z_2 - z_1) \leq z_2 - z_1$, and therefore

$$s_2 = z_1 + w_2 (z_2 - z_1) \leq z_1 + (z_2 - z_1) = z_2 = \max\{z_1, z_2\}$$

This shows that $s_2 \leq \max\{z_1, z_2\}$. Therefore, we have proved

$$\min\{z_1, z_2\} \leq s_2 \leq \max\{z_1, z_2\}$$

We see that the inequality (15) holds for $n = 2$ for the case 1.

b) Case 2: if $z_2 < z_1$, then $\min\{z_1, z_2\} = z_2$ and $\max\{z_1, z_2\} = z_1$. We have

$$\begin{aligned} s_2 &= w_1 z_1 + w_2 z_2 = (z_2 - z_2) + w_1 z_1 + w_2 z_2 \\ &= z_2 - (1 - w_2) z_2 + w_1 z_1 = z_2 - w_1 z_2 + w_1 z_1 \\ &= z_2 + w_1 (z_1 - z_2) \geq \min\{z_1, z_2\} \end{aligned}$$

This last inequality comes from the fact that $w_1 \geq 0$, and $z_1 - z_2 \geq 0$ because $\min\{z_1, z_2\} = z_2$. So we have proved $\min\{z_1, z_2\} \leq s_2$.

Because $w_1 \in [0,1]$, we have $w_1 (z_1 - z_2) \leq z_1 - z_2$, and therefore

$$s_2 = z_2 + w_1(z_1 - z_2) \le z_2 + z_1 - z_2 = z_1 = \max\{z_1, z_2\}$$

This shows that $s_2 \le \max\{z_1, z_2\}$. Therefore, we have proved

$$\min\{z_1, z_2\} \le s_2 \le \max\{z_1, z_2\}$$

We see that the inequality (15) holds for $n=2$ for the case 2. Finally, the inequality (15) is always valid for $n=2$ in all cases, i.e. when $z_1 = z_2$, or $z_1 < z_2$, or $z_2 < z_1$.

For $n>2$, we suppose that the inequality (15) holds. That is

$$\min\{z_1, ..., z_n\} \le s_n \le \max\{z_1, ..., z_n\} \tag{16}$$

We prove next by induction that this inequality also holds for $n+1$.

For the induction with $n+1$, we have to consider $n+1$ values $\{z_1, z_2, ..., z_n, z_{n+1}\}$ and $n+1$ strictly positive normalized weights $\{w_1, w_2, ..., w_n, w_{n+1}\}$, that is $w_i > 0$ for $i = 1,2,...,n$ and $\sum_{i=1}^{n+1} w_i = 1$. Because all $w_i > 0$, one has $w_{n+1} < 1$. So, we can always express s_{n+1} as

$$\begin{aligned} s_{n+1} &= \sum_{i=1}^{n+1} w_i z_i \\ &= w_{n+1} z_{n+1} + \sum_{i=1}^{n} w_i z_i \\ &= w_{n+1} z_{n+1} + \sum_{i=1}^{n} \frac{1-w_{n+1}}{1-w_{n+1}} w_i z_i \\ &= w_{n+1} z_{n+1} + (1-w_{n+1}) \sum_{i=1}^{n} \frac{w_i}{1-w_{n+1}} z_i \\ &= w_{n+1} z_{n+1} + (1-w_{n+1}) s_n \end{aligned}$$

where

$$s_n = \sum_{i=1}^{n} \frac{w_i}{1-w_{n+1}} z_i = \sum_{i=1}^{n} v_i z_i \tag{17}$$

The new weights involved in s_n defined by $v_i = \frac{w_i}{1-w_{n+1}}$ are also all strictly positive because $w_i > 0$ and $1-w_{n+1} > 0$, and they are also normalized because

$$\begin{aligned}\sum_{i=1}^{n} v_i &= \sum_{i=1}^{n} \frac{w_i}{1-w_{n+1}} \\ &= \frac{1}{1-w_{n+1}} \sum_{i=1}^{n} w_i \\ &= \frac{1}{1-w_{n+1}} (1-w_{n+1}) = 1\end{aligned}$$

because $\sum_{i=1}^{n+1} w_i = 1$, which implies $\sum_{i=1}^{n} w_i = 1-w_{n+1}$.

Hence, we observe that $s_n = \sum_{i=1}^{n} v_i z_i$ is also a convex combination of the n real values $\{z_1, z_2, ..., z_n\}$ with normalized and strictly positive weights v_i, and therefore the inequality (15) holds (by assumption).

One sees that the problem of combination of $n+1$ values has been reformulated as a convex combination of two values z_{n+1} and $s_n = \sum_{i=1}^{n} v_i z_i$ with strictly positive and normalized weights w_{n+1} and $(1-w_{n+1})$. Because the inequality (15) is satisfied for the convex combination of two real values (for $n=2$), the following inequality holds

$$\min\{s_n, z_{n+1}\} \le s_{n+1} \le \max\{s_n, z_{n+1}\} \tag{18}$$

Because $\min\{z_1, ..., z_n\} \le s_n \le \max\{z_1, ..., z_n\}$ is assumed to be true, the inequality (18) can be rewritten as

$$\min\{\min\{z_1, ..., z_n\}, z_{n+1}\} \le s_{n+1} \le \max\{\max\{z_1, ..., z_n\}, z_{n+1}\} \tag{19}$$

or equivalently

$$\min\{z_1, ..., z_n, z_{n+1}\} \le s_{n+1} \le \max\{z_1, ..., z_n, z_{n+1}\} \tag{20}$$

Therefore, the inequality (15) is also valid for $n+1$, which completes the proof of Theorem 2.

We can generalize the Theorem 2 to take into account all the cases where some weights are zero. For this, the set on n real values denoted by $\Psi=\{z_1,z_2,...,z_n\}$ can always be expressed as $\Psi=Z\bigcup\bar{Z}$, where $Z=\{z_i\in\{z_1,z_2,...,z_n\}\,|\,w_i>0\}$ and $\bar{Z}=\{z_i\in\{z_1,z_2,...,z_n\}\,|\,w_i=0\}$. The convex combination $s_n=\sum_{i=1}^{n}w_iz_i$ can be expressed as

$$s_n=\sum_{z_i\in Z}w_iz_i+\sum_{z_i\in\bar{Z}}w_iz_i \tag{21}$$

because $w_i=0$ for any $z_i\in\bar{Z}$, one has $\sum_{z_i\in\bar{Z}}w_iz_i=0$, and therefore $s_n=\sum_{z_i\in Z}w_iz_i$, whose bounds are given by Theorem 2. Hence, $\min\{z_i\in Z\}\le s_n\le\max\{z_i\in Z\}$. This completes the proof of Theorem 1, which is more general than Theorem 2.

References

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About the Authors

See the information on the last page of the first article of the authors in this issue, https://doi.org/10.11610/isij.5202, p. 36.